
Example 7.4

Solve the system of equations

$$\begin{aligned} 2x - 2y &= 6 \\ 4x + y &= 7. \end{aligned}$$

Solution

We label the equations (e_1) and (e_2) so that we can refer to them:

$$2x - 2y = 6 \tag{e_1}$$

$$4x + y = 7. \tag{e_2}$$

We can eliminate x from (e_2) by subtracting from it twice (e_1). This gives the two equations

$$2x - 2y = 6 \tag{e_1}$$

$$5y = -5. \tag{$e_2 - 2e_1$}$$

Dividing the second of these equations by 5 gives $y = -1$. Substitution of $y = -1$ into (e_1) gives $2x = 4$ so that $x = 2$.

This method essentially consists of applying certain ‘allowed’ operations to the equations within the system to produce another system having the same solution but which is easier to solve. Two or more systems of equations having the same solution(s) are called **equivalent** systems. The following operations on a system of linear equations are ‘permitted’ in that they produce another system which is equivalent:

(a) interchanging two equations;
(b) multiplying (or dividing) one equation by a non-zero constant;
(c) adding to one equation a multiple of another equation.

These operations are precisely the elementary row operations (§6.4) but applied to the equations in a system rather than the rows of a matrix.

Now suppose that we have the following linear system to solve:

$$\begin{aligned} x + y + z &= 2 \\ 2x - 2y - z &= 2 \\ 3x + y - 2z &= -2. \end{aligned}$$

We represent the system by the following partitioned matrix:

$$\left(\begin{array}{rrr|r} 1 & 1 & 1 & 2 \\ 2 & -2 & -1 & 2 \\ 3 & 1 & -2 & -2 \end{array}\right).$$

This is the matrix of coefficients which we denoted by A but with an extra column consisting of the constant terms on the right-hand sides of the equations. It is called the **augmented matrix** (or the **augmented matrix of coefficients**) and is denoted by $(A\ \boldsymbol{b})$. Each equation in the system can be reconstructed from the augmented matrix.

Each permitted operation on the equations of the system corresponds to an elementary row operation on the augmented matrix. Each elementary row operation produces a row-equivalent matrix representing an equivalent system of equations. In other words, row-equivalent augmented matrices represent equivalent systems of equations. Now suppose that we can reduce the matrix to reduced row echelon form using elementary row operations. The result might be something like

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right)$$

where a, b and c are constants. This matrix represents the system $x = a$, $y = b$, $z = c$, i.e. the solution of the original system.

This suggests a useful method for solving a system of linear equations. We write down the augmented matrix and, by applying elementary row operations to it, we reduce it (if possible) to reduced row echelon form. The solution(s) of the system can then be read off from this matrix. This process is called **Gauss–Jordan elimination**. Note that the method does not depend on being able to invert any matrix nor on the number of equations or variables. Hence it is general enough to apply to any system of linear equations.

Example 7.5

Solve the following system of linear equations:

$$\begin{aligned} x + y + z &= 2 \\ 2x - 2y - z &= 2 \\ 3x + y - 2z &= -2. \end{aligned}$$

Solution

Starting from the augmented matrix we reduce it by elementary row operations as follows:

$$\left(\begin{array}{rrr|r} 1 & 1 & 1 & 2 \\ 2 & -2 & -1 & 2 \\ 3 & 1 & -2 & -2 \end{array}\right) \qquad \text{(augmented matrix)}$$

$$\sim \left(\begin{array}{rrr|r} 1 & 1 & 1 & 2 \\ 0 & -4 & -3 & -2 \\ 0 & -2 & -5 & -8 \end{array}\right) \qquad \begin{array}{l} [R_2 \to (R_2 - 2R_1)] \\ [R_3 \to (R_3 - 3R_1)] \end{array}$$

$$\sim \left(\begin{array}{rrr|r} 1 & 1 & 1 & 2 \\ 0 & 1 & \frac{3}{4} & \frac{1}{2} \\ 0 & -2 & -5 & -8 \end{array}\right) \qquad [R_2 \to (R_2 \div (-4))]$$

$$\sim \left(\begin{array}{rrr|r} 1 & 0 & \frac{1}{4} & \frac{3}{2} \\ 0 & 1 & \frac{3}{4} & \frac{1}{2} \\ 0 & 0 & -\frac{7}{2} & -7 \end{array}\right) \qquad \begin{array}{l} [R_1 \to (R_1 - R_2)] \\ \\ [R_3 \to (R_3 + 2R_2)] \end{array}$$

$$\sim \left(\begin{array}{rrr|r} 1 & 0 & \frac{1}{4} & \frac{3}{2} \\ 0 & 1 & \frac{3}{4} & \frac{1}{2} \\ 0 & 0 & 1 & 2 \end{array}\right) \qquad [R_3 \to (R_3 \div (-\tfrac{7}{2}))]$$

$$\sim \left(\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right) \qquad \begin{array}{l} [R_1 \to (R_1 - \tfrac{1}{4}R_3)] \\ [R_2 \to (R_2 - \tfrac{3}{4}R_3)]. \end{array}$$

Thus the solution is $x = 1$, $y = -1$, $z = 2$. These can be checked by substitution into the three equations.

The systematic steps required to reduce the augmented matrix to reduced row echelon form are exactly those which we employed to find the inverse of a matrix A when we used elementary row operations to convert the partitioned matrix $(A\ I)$ to $(I\ A^{-1})$. The sequence of steps is given on page 319 and illustrated in the flowchart in figure 6.1. In fact, if A (the matrix of coefficients) is a square non-singular matrix, the process of Gauss–Jordan elimination will inevitably result in a reduced row echelon form which consists of the appropriate identity matrix with an extra column on the right-hand side.

We now see what happens if A does not have an inverse, either because (a) A is a square singular matrix, or (b) A is not a square matrix.