zero. On the other hand, the set of all real functions has unit measure (cf (1.1.55)). From this fact we immediately derive the fact that the set of *continuous* functions has unit measure, and, moreover, (1.1.72) shows that functions satisfying the Hölder–Lipschitz condition (see, e.g., Ilyin and Poznyak (1982))

$$|x(t_2) - x(t_1)| < h|t_2 - t_1|^{\frac{1}{2}-\varepsilon} \tag{1.1.73}$$

also have unit measure (we shall use this fact later, in section 1.2.7).

The proof of Wiener's theorem for the case of differentiable trajectories is considered in problem 1.1.5, page 53.

◇ Integration of functionals: general approach

So far, we have discussed mainly the Wiener measure of sets of trajectories. To develop the integration theory further, we begin by considering *simple functionals* $F[x(\tau)]$ for which the path integrals of the form

$$\int_{\mathcal{C}\{0,0;t\}} d_{\mathrm{W}}x(\tau)\, F[x(\tau)] \equiv \int_{\mathcal{C}\{0,0;t\}} \prod_{\tau=0}^{t} \frac{dx(\tau)}{\sqrt{4\pi D\, d\tau}} \mathrm{e}^{-\frac{1}{4D}\int_0^t \dot{x}^2(\tau)\, d\tau} F[x(\tau)] \tag{1.1.74}$$

can be evaluated immediately. Measurable functionals will be defined as appropriate infinite limits of the simple functionals. The proof of the Wiener theorem prompts the obvious example of a simple functional: this is the characteristic functional of a measurable set. If we take

$$F[x(\tau)] = \chi_Y[x(\tau)]$$

where χ_Y is the characteristic functional of some set Y of the trajectories $x(\tau)$ defined as

$$\chi_Y[x(\tau)] = \begin{cases} 1 & \text{if } x(\tau) \in Y \\ 0 & \text{if } x(\tau) \notin Y \end{cases}$$

then the definition gives

$$\int d_{\mathrm{W}}x(\tau)\, \chi_Y[x(\tau)] = \mu_{\mathrm{W}}(Y)$$

where $\mu_{\mathrm{W}}(Y)$ is the Wiener measure of the set Y (cf explanation in the proof of Wiener's theorem, below equation (1.1.66)).

As an example, let us choose the set $Y(x_1, \ldots, x_N)$ of the trajectories having fixed positions $x_1, \ldots, x_N$ at some sequence $t_1, \ldots, t_N$ of the time variable:

$$\chi_Y[x(\tau); x_1, \ldots, x_N] = \begin{cases} 1 & \text{if } x(t_1) = x_1, \ldots, x(t_N) = x_N \\ 0 & \text{otherwise.} \end{cases}$$

The corresponding path integral is given by the product of the transition probabilities (cf (1.1.50) and (1.1.51)) from the points $x(t_{i-1})$ to the next positions $x(t_i)$ in the sequence:

$$\begin{aligned} \int_{\mathcal{C}\{0,0;t\}} d_{\mathrm{W}}x(\tau)\, \chi_Y[x(\tau); x_1, \ldots, x_N] &= \prod_{i=1}^{N} W(x_i, t_i | x_{i-1}, t_{i-1}) \\ &= \prod_{i=1}^{N} \frac{1}{\sqrt{4\pi D(t_i - t_{i-1})}} \exp\left\{ -\frac{(x_i - x_{i-1})^2}{4D(t_i - t_{i-1})} \right\} \\ &\quad x_0 = 0,\ t_0 = 0. \end{aligned} \tag{1.1.75}$$

Now we can consider linear combinations of such characteristic functionals with some coefficients $F_N(x_1, \ldots, x_N)$,

$$F_N[x(\tau)] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dx_1 \cdots dx_N \, F_N(x_1, \ldots, x_N)\chi_Y[x(\tau); x_1, \ldots, x_N]. \tag{1.1.76}$$

Making use of (1.1.75), the functional integration of these functionals can be easily reduced to the ordinary finite-dimensional integration

$$\int_{\mathcal{C}\{0,0;t\}} d_{\mathrm{W}}x(\tau) \, F_N[x(\tau)] = \int_{-\infty}^{\infty} \frac{dx_1}{\sqrt{4\pi Dt_1}} \cdots \frac{dx_N}{\sqrt{4\pi D(t_N - t_{N-1})}} \\ \times F_N(x_1, \ldots, x_N) \exp\left\{ -\frac{1}{4D} \sum_{i=1}^{N} \frac{(x_i - x_{i-1})^2}{t_i - t_{i-1}} \right\}. \tag{1.1.77}$$

We see that $F_N(x_1, \ldots, x_N)$ should be such a function of $x_1, \ldots, x_N$ that the right-hand side of (1.1.77) exists, e.g., it can be a polynomial.

These *simple functionals* form a vector space $\mathcal{F}$ and the Wiener path integral allows us to define the norm (distance) $\|\cdot\|$ in this vector space:

$$\|F - G\| = \int d_{\mathrm{W}}\, x(\tau)|F - G| \tag{1.1.78}$$

($|\cdot|$ on the right-hand side is just the absolute value of the difference of the two functionals), where F and G are two simple functionals (i.e. of the type (1.1.76)). Readers with mathematical orientation may easily check that all the axioms for a norm are satisfied by (1.1.78). Having defined the norm (1.1.78), the general problem of extending the path integral to a larger set of functionals can be accomplished by the standard mathematical method. First, we define a sequence $F^{(N)}$ of simple functionals with the property

$$\|F^{(N)} - F^{(M)}\| \xrightarrow[N,M\to\infty]{} 0$$

called the *Cauchy sequence*. The functional $F[x(\tau)]$ is said to be *integrable* if there exists a Cauchy sequence $F^{(N)}$ of simple functionals such that

$$F^{(N)} \xrightarrow[N\to\infty]{} F \tag{1.1.79}$$

with respect to the norm (1.1.78). The path integral is then defined by

$$\int_{\mathcal{C}\{0,0;t\}} d_{\mathrm{W}}x(\tau) \, F[x(\tau)] \stackrel{\text{def}}{\equiv} \lim_{N\to\infty} \int_{\mathcal{C}\{0,0;t\}} d_{\mathrm{W}}x(\tau) \, F_N[x(\tau)]. \tag{1.1.80}$$

◇ Practical method of integration of functionals: approximation by piecewise linear functions

In practice, we can use the fact that the set of functions possessing non-zero Wiener measure can be uniformly approximated by piecewise linear functions $\ell_N(\tau)$ which are linear for $\tau \in (t_{i-1}, t_i)$ and

$$\ell_N(t_i) = x(t_i) \equiv x_i \qquad i = 1, \ldots, N$$

(see figure 1.8). This means that for any ε and any function $x(t)$, there exist points $t_0, t_1, \ldots, t_N$; $N = N(\varepsilon)$ such that

$$|x(\tau) - l_N(\tau)| < \varepsilon \qquad N = N(\varepsilon).$$